AMSAR/SAO/N-07

GENERAL STATISTICAL PROCEDURES: PARAMETER ESTIMATION USING WEIBULL DISTRIBUTION, RELIABILITY TEST OF HYPOTHESIS, AND COMPUTATION OF EXPECTED NUMBER OF RENEWALS

THOMAS N. MAZZA
ROBERT C. BANASH

AUGUST 1975

US ARMY ARMAMENT COMMAND
SYSTEMS ANALYSIS OFFICE
ROCK ISLAND, ILLINOIS 61201

DISPOSITION

Destroy this report when no longer needed. Do not return it to the originator.

DISCLAIMER

The findings in this report are not to be construed as an official Department of the Army position.

WARNING

Information and data contained in this document are based on input available at the time of preparation. Because the results may be subject to change, this document should not be construed to represent the official position of the US Army Materiel Command unless so stated.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER AMSAR/SAO/N-07	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) General Statistical Procedures: Parameter Estimation Using Weibull Distribution, Reliability Test of Hypothesis, and Computation of Expected Number of Renewals.		5. TYPE OF REPORT & PERIOD COVERED Note - Final April 1973 6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Thomas N. Mazza Robert C. Banash		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS US Army Armament Command Systems Analysis Office Rock Island, IL 61201		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS US Army Armament Command Systems Analysis Office Rock Island, IL 61201		12. REPORT DATE August 1975 13. NUMBER OF PAGES 30
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

This note is an addendum to PAA-TR-1-73, April 1973, AD 763204.

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Statistics
Maximum likelihood
Weibull Distribution
Reliability
Durability
Renewal theory
Renewal rate
Risk Analysis
Probability theory

20. ABSTRACT (Continue on reverse side if necessary and identify by block number) This addendum details the statistical procedures utilized in preparing the report, Decision Risk Analysis for XM204, 105mm Howitzer, Towed Reliability/Durability Requirements (PAA-TR-1-73). A computer program was developed for this study to simulate DT/OT II testing analysis of data, the decision made based on this analysis, and the funding implications of this decision. A two-parameter Weibull family was assumed to describe durability failures. The test results were used to estimate the two parameters for each subsystem: carriage, recoil system, tube, and breech. This was followed by a test of hypothesis which tested whether sufficient information was

20. ABSTRACT

available to reject the hypothesis that the required durability had been obtained. The subsystem durability was then used to compute replacement requirements over the system lifetime.

CONTENTS

Next page is blank.

INTRODUCTION

This addendum details the statistical procedures utilized in preparing the report, Decision Risk Analysis for XM204, 105MM Howitzer, Towed Reliability/Durability Requirements (PAA-TR-1-73).[1] A computer program was developed for this study to simulate DT/OT II testing, analysis of test data, the decision made based on this analysis, and the funding implications of this decision.

A two-parameter Weibull family was assumed to describe durability failures. The test results were used to estimate the two parameters for each subsystem: carriage, recoil system, tube, and breech. This was followed by a test of hypothesis which tested whether sufficient information was available to reject the hypothesis that the required durability had been obtained. The subsystem durability was then used to compute replacement requirements over the system lifetime.

PROBLEMS

The first statistical problem, parameter estimation, was unusual in that a large number of subsystems would survive testing. This occurred as the carriage was a long-lived subsystem and the test truncation point was near the expected carriage life. Also, other failed subsystems would be replaced until one of the following occurred:

a. The carriage failed.

b. The truncation point was obtained.

c. No additional back-up subsystems were available.

Thus, for each subsystem the test data would consist of rounds-to-failure

[1] Mazza, Thomas N. and Banash, Robert C., PAA-TR-1-73. Decision Risk Analysis for XM204, 105MM Howitzer, Towed Reliability/Durability Requirements, Systems Analysis Division, Plans and Analysis Directorate, US Army Weapons Command, April 1973.

and random truncation points. Parameter estimation was performed by maximizing the log likelihood equations.

The next problem was to use these data in a test of hypothesis. This was accomplished by using the large-sample distribution properties of maximum likelihood estimaters, i.e., that the parameters have a joint normal distribution. The α-percentile confidence interval is approximately ellipsoidal; the α-percentile durability boundary was obtained by maximizing over this ellipse.

The third problem was solving for the number of renewals in a specified time interval. This was accomplished, initially, by the method of Lomnicki[2] and later (to save computer time), by taking averages.

The purpose of this addendum is to detail these procedures.

PARAMETER ESTIMATION

The form of the density function for the two-parameter Weibull distribution used in this study is:

$$f(t) = \lambda \alpha t^{\alpha-1} e^{-\lambda t^{\alpha}} dt \qquad t \geq 0; \alpha > 0; \lambda > 0 . \tag{1}$$

Consider the following subsystem-life test. M-like systems are placed on test and each system is composed on one critical subsystem and several different noncritical subsystems. A total system configuration is required to conduct the test; however, with respect to probability of failure, each subsystem is assumed to be independent. As each non-critical subsystem failure occurs, the failure time is noted, and the failed subsystem

[2] Lomnicki, Z. A., A Note on the Weibull Process. Biometrika 53, 375-381 (1966).

is replaced with an identical new subsystem. Each of the M systems continue with the test, with failed noncritical subsystems being replaced, until either a predetermined system truncation point (for example, when a stated number of rounds have been fired) is reached or until a critical subsystem failure occurs. This test differs from the well-known Type I and Type II censoring in that the subsystem truncation points are random variables as is the number of each type of noncritical subsystems put on test.

Let N be the total number of a particular noncritical subsystem put on test. Let n of these subsystems fail and their failure times be observed. The remaining m = N - n subsystems are removed from test at the truncation points $T_1, T_2, T_3, \ldots T_m$. Then the logarithmic likelihood function $\ln L(\alpha,\lambda)$, based on the above sample where (1) is the applicable failure density function, is given by

$$\ln L(\alpha,\lambda) = n\ln\alpha + n\ln\lambda + (\alpha-1)\sum_{i=1}^{n}\ln y_i - \lambda\sum_{i=1}^{n} y_i^{\alpha} - \lambda\sum_{j=1}^{m} x_j^{\alpha} \quad (2)$$

where

y_i = an observed failure time,

and

x_j = a system truncation point.

Then

$$\ln L(\alpha,\lambda) = n\ln\alpha + n\ln\lambda + (\alpha-1)\sum_{i=1}^{n}\ln t_i - \lambda\sum_{i=1}^{N} t_i^{\alpha} \quad (3)$$

where

$$t_i = y_i \qquad i \leq n$$

$$= x_i \qquad n < i \leq N.$$

This function yields the following likelihood equations

$$\frac{\partial \ln L}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \ln t_i - \lambda \sum_{i-1}^{N} t_i^{\alpha} \ln t_i = 0 \ , \tag{4}$$

and

$$\frac{\partial \ln L}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{N} t_i^{\alpha} = 0 \ . \tag{5}$$

Solving (5) in terms of λ and substituting into (4) yields

$$\frac{n}{\alpha} + \sum_{i=1}^{n} \ln t_i - \frac{n}{\sum_{i=1}^{N} t_i^{\alpha}} \cdot \sum_{i=1}^{N} t_i^{\alpha} \ln t_i = 0 \ . \tag{6}$$

Equation (6) can be solved for by a standard iterative procedure but by writing (6) in the form

$$K(\alpha) = \frac{\sum_{i=1}^{N} t_i^{\alpha} \ln t_i}{\sum_{i=1}^{N} t_i^{\alpha}} - \frac{1}{\alpha} - \frac{\sum_{i=1}^{n} \ln t_i}{n} = 0 \ , \tag{7}$$

it has been shown[3] that if we find a $K(\alpha_1) < 0$ and a $K(\alpha_2) > 0$ and α_1 and α_2 are within a sufficiently narrow interval such that $\alpha_1 < \hat{\alpha} < \alpha_2$, a linear interpolation will yield the required value. While this method will eventually yield an answer, it would appear that the rate of convergence can be immensely improved by utilizing basic search techniques. In addition, the search space can be reduced by choosing an initial α_1 that is relatively close to $\hat{\alpha}$. Dubey[4] has suggested that the moment estimators be utilized for the initial value and Cohen[3] developed a table of values relating α and the sample coefficient of variation. Tables relating the coefficient of variation to α require external storage areas (disks/tapes/drums) or, if internal to a computer program, take up precious core storage. Therefore, the following equation is utilized which yields an initial estimator based on the coefficient of variation which is within 0.1% of similar table values.

$$\begin{aligned}\alpha_1 = {} & .64364 + .18035 \operatorname{csch}(z) - .0317523z + .000684128z^2 \\ & - .00129259 \operatorname{csch}^2(z) + .619344(e^{-z}) + .534717(e^{-z})^4 \\ & + .54047(e^{-z})^7\end{aligned} \tag{8}$$

where

$$z = \frac{s^2}{\bar{x}^2} = cv = \text{coefficient of variation.}$$

With this initial estimator, a "direct search with acceleration" is utilized to obtain the maximum likelihood estimate (MLE) of α or $\hat{\alpha}$.

[3]Cohen, A. C., Maximum Likelihood Estimation in the Weibull Distribution Based on Complete and on Censored Samples. Technometrics 7 (4), 579-588 (1965).

[4]Dubey, S. A., On Some Statistical Inferences for Weibull Laws, Naval Research Logistics Quarterly 13, 227-251 (1966).

It has been shown[5] that $\frac{\hat{\alpha}}{\alpha}$ is distributed independently of α and λ and has the same distribution as $\hat{\alpha}$. Therefore, the bias in $\hat{\alpha}$ is independent of the true value of α and λ and is only dependent on the sample size n. A table of unbiasing factors B(n) was developed so that $E[B(n)\hat{\alpha}] = \alpha$. The values in this table are approximated by the following equation:

$$B(N) = .06541717052 - .00006172777867n + .997393979 \tanh^{10}(n)$$
$$-220.5624312 \tanh^{7}(\ln n/n) - 1.86171021(\ln n)(\tanh(\ln n/n^{2})) \qquad (9)$$
$$+.01424021769(\ln n) \tanh^{10}(n) - 39302536740000.\tanh^{8}(\ln n/n^{3})$$

Once the MLE of α is obtained from the search of equation (7), the appropriate unbiasing factor equation (9) is applied before solving for λ, using equation (5).

AN ILLUSTRATIVE EXAMPLE

Ten systems are placed on test and the truncation point, T_i, for each system is given (in rounds) below.

T_1	-	35,142	T_6	-	76,958
T_2	-	43,839	T_7	-	78,962
T_3	-	45,540	T_8	-	79,864
T_4	-	61,471	T_9	-	91,190
T_5	-	66,681	T_{10}	-	95,202

During the course of the test, 23 failures/replacement (Y_i) of subsystem z occurred at the following time:

[5]Thoman, D. R., Bain, L. J., and Antle, C. E., Inferences on the Parameters of the Weibull Distribution. Technometrics 11 (3), 445-460 (1969).

y_1	-	33,748	y_9	-	36,102	y_{17}	-	14,326
y_2	-	24,898	y_{10}	-	6,822	y_{18}	-	27,959
y_3	-	10,376	y_{11}	-	28,824	y_{19}	-	10,128
y_4	-	27,186	y_{12}	-	37,452	y_{20}	-	31,864
y_5	-	34,377	y_{13}	-	1,170	y_{21}	-	9,366
y_6	-	13,591	y_{14}	-	39,886	y_{22}	-	21,978
y_7	-	878	y_{15}	-	10,212	y_{23}	-	12,924
y_8	-	41,524	y_{16}	-	24,029			

and at the time of each system truncation, the following subsystem times (x_i) were observed:

x_1	-	1,395	x_6	-	40,855
x_2	-	8,564	x_7	-	5,863
x_3	-	18,354	x_8	-	38,806
x_4	-	12,624	x_9	-	14,662
x_5	-	25,157	x_{10}	-	8,940

This sample for subsystem z is from a population in which $\alpha = 1.364$ and $\lambda = .00000102336$. This data can be summarized as:

$$n=23,\ N=33,\ \sum_1^{23} t_i=499618,\ \sum_1^{23} t_i^2=14396873864.$$

It follows that $\bar{x} = 21722.52$, $s^2= 154083086.858$, and the coefficient of variation cv = .3265. Then from (8) the initial estimate of α=1.81. To obtain the maximum likelihood estimate, α is varied in equation (7) until $|K(\alpha)|$ is minimized. This results in $\hat{\alpha}$=1.6421 and applying the unbiasing factor for a sample size of 33, $\hat{\alpha} = 1.5731$.

TEST OF HYPOTHESIS

Consider the following failure density function:

$$f(t;\alpha,\lambda) = \alpha\lambda t^{\alpha-1} e^{-\lambda t^{\alpha}} dt \quad \alpha>0;\lambda>0;t\geq 0 \ . \tag{10}$$

When the resulting sample likelihood function of a life test can be written as

$$L = \prod_{i=1}^{n} \left[\alpha\lambda t_i^{\alpha-1} e^{-\lambda t_i^{\alpha}} \right] e^{-\lambda \sum_{j=n+1}^{N} t_j^{\alpha}} \tag{11}$$

then, this function yields the following partial derivatives of the logarithmic likelihood equations:

$$\frac{\partial \ln L}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \ln t_i - \lambda \sum_{i=1}^{N} t_i^{\alpha} \ln t_i = 0 \ , \tag{12}$$

$$\frac{\partial \ln L}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{N} t_i^{\alpha} = 0 \ , \tag{13}$$

$$\frac{\partial^2 \ln L}{\partial \alpha^2} = \frac{-n}{\alpha^2} - \lambda \sum_{i=1}^{N} t_i^{\alpha} (\ln t_i)^2 = 0 \ , \tag{14}$$

$$\frac{\partial^2 \ln L}{\partial \lambda^2} = \frac{-n}{\lambda^2} = 0 \ , \tag{15}$$

$$\frac{\partial \ln L}{\partial \alpha \partial \lambda} = - \sum_{i=1}^{N} t_i^{\alpha} \ln t_i = 0 \ . \tag{16}$$

The maximum likelihood estimate (MLE) $\hat{\alpha}$ and $\hat{\lambda}$ can be obtained by solving (13) in terms of λ and substituting into (12). Having obtained the MLE for the parameters of the Weibull law, it is then possible to determine confidence limits for meaningful parametric functions such as durability and reliability.

Durability can be defined as the probability that a randomly selected item from an infinite lot will continue to perform satisfactorily without a durability failure beyond t_0 and is given by

$$\mathrm{Prob}[T \geq t_0] = \int_{t_0}^{\infty} f_T(t)dt = \exp(-\lambda t_0^{\alpha}) \ . \tag{17}$$

If the parameters α and λ are known, the above problem is completely solved and the exact answer is given by (17); however, usually these parameters are unknown. It is well known that the large sample MLE is approximately normally distributed about the true parameter value as a mean for large samples. This is a powerful tool and will be used to establish confidence limits on durability when the true parameters α and λ are unknown. Assuming the above, it follows that[6]

$$\begin{bmatrix} \hat{\alpha}-\alpha, \hat{\lambda}-\lambda \end{bmatrix} \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} \hat{\alpha}-\alpha \\ \hat{\lambda}-\lambda \end{bmatrix} \simeq \chi^2(2) \ , \tag{18}$$

where

$$C_{11} = -E \left[\frac{\partial^2 \ln L}{\partial \alpha^2} \right] , \tag{19}$$

[6] Mood, A. M., Introduction to the Theory of Statistics. McGraw-Hill Book Co., Inc., New York, New York, 1950.

$$C_{22} = -E\left[\frac{\partial^2 \ln L}{\partial \lambda^2}\right] , \qquad (20)$$

and

$$C_{12} = C_{21} = -E\left[\frac{\partial^2 \ln L}{\partial \alpha \partial \lambda}\right] . \qquad (21)$$

From (14) and (19)

$$C_{11} = E\left[\frac{n}{\alpha^2} + \lambda \sum_{i=1}^{N} t_i^{\alpha}(\ln t_i)^2\right] . \qquad (22)$$

Examine $$I = \int_0^\infty t^\alpha (\ln t)^2 \alpha \lambda t^{\alpha-1} e^{-\lambda t^\alpha} dt$$

and let

$$\omega = t^\alpha ,$$

$$d\omega = \alpha t^{\alpha-1} dt ,$$

$$\ln \omega = \alpha \ln t ,$$

$$\ln t = \frac{1}{\alpha} \ln \omega ,$$

$$(\ln t)^2 = \frac{1}{\alpha^2} (\ln \omega)^2 .$$

then

$$I = \frac{\lambda}{\alpha^2} \int_0^\infty \omega (\ln \omega)^2 \, e^{-\lambda \omega} \, d\omega .$$

From page 578 of referenced literature[7], equation 4.358(2)

---

[7]Gradshteyn, I.S., and Ryzhik, I.M., Table of Integrals, Series, and Products. Academic Press, New York, New York. 1965.

$$\int_0^\infty x^{v-1}e^{\mu x}(\ln x)^2 dx = \Gamma(v)/\mu^v\{[\psi(v)-\ln(\mu)]^2+ \zeta\ (2,v-1)\}$$

$$\mu > 0, v > 0$$

now let

$$x = \omega,$$

$$v = 2,$$

and

$$\mu = \lambda,$$

then

$$I = \frac{\lambda}{\alpha^2}\int_0^\infty \omega(\ln\omega)^2 e^{-\lambda\omega}d\omega = \lambda\Gamma(2)/(\alpha^2\lambda^2)\{[\psi(2)-\ln\lambda]^2+\zeta(2,1)\}$$

$$= \frac{\Gamma(2)}{\lambda\alpha^2}\{[\psi(2)+\ln\frac{1}{\lambda}]^2 + \zeta(2,1)\}$$

From page 1073 of referenced literature[8], equation 9.521(1) the Riemann's Zeta Function

$$\zeta(z,q) = \sum_{N=0}^{\infty} \frac{1}{(q+N)^2} \qquad z > 1$$

$$\therefore\ \zeta(2,1) = \frac{1}{1} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \ldots \quad .$$

From page 260 of referenced literature[8], equation 6.4.10 the Polygamma Function

$$\psi^N(z)=(-1)^{N+1}N!\sum_{k=0}^{\infty}(z+k)^{-N-1} \qquad (z\neq 0,-1,-2,\ldots)$$

[8]Abramowitz, M., and Stegun, I. A., Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, National Bureau of Standards. US Government Printing Office, Washington, DC. 1970

$$\therefore \quad \psi^1(1) = (-1)\ 1! \sum_{k=0}^{\infty} (1+k)^{-2} = \frac{1}{1} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots ,$$

therefore,

$$\zeta(2,1) = \psi^1(1)$$

and

$$I = \frac{\Gamma(2)}{\lambda\alpha^2} \{[\psi(2)+\ln \tfrac{1}{\lambda}]^2 + \psi^1(1)\} .$$

Equation (22) can now be written as

$$C_{11} = \frac{n}{\alpha^2} + \frac{N}{\alpha^2} \{[\psi(2)+\ln \frac{1}{\lambda}]^2 + \psi^1(1)\} \qquad (23)$$

where

$$\psi(2) = .4227843351 ,$$

and

$$\psi^1(1) = 1.6449340668 .$$

From (16) and (21)

$$C_{12} = E\left[\sum_{i=1}^{N} t_i^{\alpha} \ln t_i\right] \qquad (24)$$

Examine

$$I = \int_0^{\infty} t^{\alpha} \ln t \ \alpha\lambda t^{\alpha-1}\ e^{-\lambda t^{\alpha}} dt$$

and let

$$\omega = t^{\alpha} ,$$

$$d\omega = \alpha t^{\alpha-1} dt ,$$

$$\ln\omega = \alpha \ln t ,$$

$$\ln t = \ln\omega/\alpha ,$$

then

$$I = \frac{\lambda}{\alpha} \int_0^\infty \omega \ln\omega \; e^{-\lambda\omega} \, d\omega \quad .$$

From page 576 of referenced literature[7], equation 4.352(1)

$$\int_0^\infty x^{v-1} \; e^{-\mu x} \; \ln x dx = \frac{1}{\mu^v} \Gamma(v)[\psi(v)-\ln(\mu)] \qquad u > 0, v > 0$$

let

$$v = 2 \; ,$$

$$\mu = \lambda \; ,$$

$$x = \omega \; ,$$

then

$$I = \frac{\lambda}{\alpha} \int_0^\infty \omega \ln\omega \; e^{-\lambda\omega} \, d\omega = \left(\frac{\lambda}{\alpha}\right)\left(\frac{1}{\lambda^2}\right)\Gamma(2)[\psi(2)-\ln\lambda]$$

$$= \frac{1}{\alpha\lambda} \left[\psi(2)+\ln \frac{1}{\lambda}\right] \; .$$

Equation (24) can now be written

$$C_{12} = \frac{N}{\lambda\alpha} \left[\psi(2)+\ln \frac{1}{\lambda}\right] \quad , \tag{25}$$

and it follows that

$$C_{22} = \frac{N}{\lambda^2} \quad . \tag{26}$$

As just shown, the $C_{i,j}$ matrix is the inverse asymptotic variance-covariance matrix of $(\hat{\lambda},\hat{\alpha})$ and is obtained by taking the negatives of the expected values of the second order derivatives of logarithms of the likelihood functions. Using (18), we can now obtain the appropriate confidence limits for the true probabilities. In the case of reliability

[7] Loc. Cit.

or durability, we want to insure that the true reliability or durability is above some minimum value R_m or D_m at a desired confidence level. From (18), then,

$$C_{11}(\hat{\alpha}-\alpha)^2 + 2C_{12}(\hat{\lambda}-\lambda)(\hat{\alpha}-\alpha) + C_{22}(\hat{\lambda}-\lambda)\approx\chi^2(2) \; . \tag{27}$$

The above equation describes the boundary of a confidence region in the parametric space (α,λ). It is approximately chi-squared distributed with two degrees of freedom and will serve to determine an ellipsoidal confidence region in the (α,λ) space. In order to obtain the upper confidence limit, all that is required is to search the parameter space, (27), varying α and λ until the durability function (17) is maximized.

A PROCEDURE FOR COMPUTING EXPECTED NUMBER OF RENEWALS

Let X_1, X_2, X_3,... be a renewal process, that is, a sequence of independent, nonnegative and identically distributed random variables which are not all zero with probability one, with the probability density function f(X) and the distribution function F(X). $S_k = X_1 + X_2 + \ldots X_k$ is interpreted in renewal theory as the time up to the kth renewal and the probability that $S_k < x$ is given by the k-fold convolution of F(x)

$$F_K(x) = \int_0^x F_{K-1}\,(x-t)\;d\,F(t),$$

where

$$F_0(x) = 1.$$

The primary purpose of this procedure is to calculate N_t which is defined to be the maximum suffix K for which $S_k \leq t$, subject to the convention $N_t = 0$, if $X_1 > t$. In this application, the X_i represents successive lifetimes of the object being renewed, and N_t is the number of renewals made by time t, subject to the original object having been installed at time 0, i.e., N_t is the number of renewals in $(0,t)$.

The usual procedure for determining various functions of Renewal Theory is to find the Laplace Transforms of these functions and then revert to the time domain. However, in the case of the Weibull distribution, this approach is not convenient and Smith and Leadbetter[9] have developed an expansion of the Renewal Function into a power series of t^α where α is the Weibull shape parameter. White[10] demonstrated a procedure for evaluating the higher moments and cumulants of the number of renewals N_t and computed the mean $M(t)$ and the standard deviation, $(\text{Var } N_t)^{1/2}$, of N_t. Lomnicki[2] developed a method to evaluate the distribution of the number of renewals which is defined by the family of functions $W_k(t)$ where

$$W_k(t) = F_k(t) - F_{k+1}(t) \qquad (k=0,1,\ldots)$$

and is the probability of exactly k renewals in $(0,t)$.

In order to be able to evaluate more of the various forms of the renewal functions in the case of the Weibull renewal process, the basic method developed by Lomnicki was used.

As shown by Lomnicki, if we assume that $W_k(t)$ is represented by the unique series

$$W_k(t) = \sum_{s=k}^{\infty} a_k(s) P_s(t^\alpha)$$

[9]Smith, W. L., and Leadbetter, M. R., On the Renewal Function for the Weibull Distribution. Technometrics 5(3), 393-396 (1963).

[10]White, J. S., Weibull Renewal Analysis. Third Annual Aerospace Reliability and Maintainability Conference. pp 639-657. Society of Automotive Engineers, Inc. New York, New York. 1964.

[2]Loc. Cit.

where

$$P_k(t) = e^{-t}\,\frac{t^k}{k!} \qquad (k=0,1,2,\ldots) \quad ,$$

then

$$W_k(t) = \sum_{s=k}^{\infty} P_s(t^{\alpha}) \sum_{p=k}^{s} (-1)^{p+k} \binom{s}{p} \frac{b_k(p)}{\gamma(p)} \quad ,$$

and we have

$$a_k(s) = \sum_{p=k}^{s} (-1)^{p+k} \binom{s}{p} \frac{b_k(p)}{\gamma(p)} \qquad (k=0,1,\ldots;s=k,k+1,\ldots)$$

where

$$\gamma(r) = \Gamma(\alpha r+1)/\Gamma(r+1) \qquad (r=0,1,\ldots)$$

and

$$b_0(s) = \gamma(s). \qquad (s=0,1,\ldots)$$

$$b_{k+1}(s) = \sum_{r=k}^{s-1} b_k(r)\gamma(s-r) \qquad (k=0,1,\ldots;s=k,k+1,k+2,\ldots) \; .$$

The expected number of renewals $M(t) = E[N(t)]$ which is the renewal function is given by

$$M(t) = \sum_{n=0}^{\infty} n\, P[N(t)=n]$$

$$M(t) = \sum_{n=0}^{\infty} n\, W_n(t) \quad .$$

Since

$$W_1(t) + 2W_2(t) + 3W_3(t) + \ldots$$

is equal to

$$F_1(t) - F_2(t) + 2F_2(t) - 2F_3(t) + 3F_3(t) - 3F_4(t) + \ldots ,$$

then

$$\sum_{n=0}^{\infty} n\, W_n(t) = \sum_{n=0}^{\infty} F_n(t)$$

and

$$M(t) = \sum_{n=0}^{\infty} F_n(t) \; ,$$

where $F_n(t)$ is the convolution of the cumulative distribution function $F(t)$ of the first renewal time and the n-1 subsequent renewal-time distributions and is the distribution of S_n, the time of the nth renewal. Now,

$$F_k(t) = W_k(t) + F_{k+1}(t)$$

$$= \ldots + W_{k+1}(t) + F_{k+3}(t)$$

$$\vdots$$

$$F_k(t) = \sum_{r=k}^{\infty} W_r(t) \qquad\qquad (k=1,2,3\ldots)$$

or

$$F_k(t) = 1 - \sum_{r=0}^{k-1} W_r(t),$$

so we can now express $F_k(t)$ as

$$F_k(t) = \sum_{r=k}^{\infty} \sum_{s=r}^{\infty} a_r(s)\; P_s(t^{\alpha}): \quad .$$

Substituting the Poisson cumulative function

$$D_s(t) = \sum_{r=s}^{\infty} e^{-t} \frac{t^r}{r!}$$

means that

$$F_k(t) = \sum_{s=k}^{\infty} \beta_k(s)\, D_s(t^\alpha)$$

where

$$\beta_k(k) = a_k(k) \quad ,$$

and

$$\beta_k(s) = \sum_{r=k}^{s} a_r(s) - \sum_{r=k}^{s-1} a_r(s-1) \quad (s > k) \ ,$$

so that

$$M(t) = \sum_{k=1}^{\infty} F_K(t) = \sum_{s=1}^{\infty} D_s(t^\alpha) \sum_{k=1}^{s} \beta_k(s) \ ,$$

or

$$M(t) = \sum_{k=1}^{\infty} C(s)\, D_s(t^\alpha) \quad ,$$

where

$$C(s) = \sum_{k=1}^{s} \beta_k(s) \quad . \quad (s=1,2,...) \ .$$

We can now evaluate $M_k(t)$, the Renewal Function, $W_k(t)$; the probability of exactly k renewals; and $F_k(t)$, the distribution of time of the kth renewal. These functions are very useful in evaluating models of reliability, inventory, and queueing process.

$$M_k(t) = E\binom{N(t)}{k} = \sum_{n=0}^{\infty} \binom{n}{k} (F_n(t) - F_{n+1}(t)) \ .$$

The moments of N(t) may be derived from $M_k(t)$ by the relationship

$$E(N(t)^k) = \sum_{n=1}^{k} t_{k,n}\, M_n(t) n!$$

where $t_{k,n}$ are the Stirling numbers of the second kind and represent the number of ways of partitioning a set of k elements into n nonempty subsets. In particular

$$E\{N(t)\} = M_1(t) \quad ,$$

and

$$E\{N(t)^2\} = M_1(t) + 2M_2(t) \quad .$$

Or, if desired, the moments of N(t) may be derived from the definition of expected values.

$$E\{N(t)^K\} = \sum_{n=0}^{\infty} n^K \, \mathrm{Prob}(N(t)=n) \quad ,$$

$$E\{N(t)^K\} = \sum_{n=0}^{\infty} n^K\{F_n(t)-F_{n+1}(t)\} \quad ,$$

$$E\{N(t)^K\} = \sum_{n=0}^{\infty} n^K W_n(t) \quad ,$$

$$E\{N(t)^K\} = \sum_{n=1}^{\infty} n^K-(n-1)^K F_n(t) \quad .$$

Next page is blank.

LITERATURE CITED

1. Mazza, Thomas M. and Banash, Robert C., PAA-TR-1-73. Decision Risk Analysis for XM204, 105MM Howitzer, Towed Reliability/Durability Requirements, Systems Analysis Division, Plans and Analysis Directorate, US Army Weapons Command, April 1973.

2. Lomnicki, Z. A., A Note on the Weibull Process. Biometrika 53, 375-381 (1966).

3. Cohen, A. C., Maximum Likelihood Estimation in the Weibull Distribution Based on Complete and on Censored Samples. Technometrics 7 (4), 579-588 (1965).

4. Dubey, S. A., On Some Statistical Inferences for Weibull Laws, Naval Research Logistics Quarterly 13, 227-251 (1966).

5. Thoman, D. R., Bain, L. J., and Antle, C. E., Inferences on the Parameters of the Weibull Distribution. Technometrics 11 (3), 445-460 (1969).

6. Mood, A. M., Introduction to the Theory of Statistics. McGraw-Hill Book Co., Inc. New York, New York, 1950.

7. Gradshteyn, I. S., and Ryzhik, I. M., Table of Integrals, Series, and Products. Academic Press, New York, New York. 1965.

8. Abramowitz, M., and Stegun, I. A., Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, National Bureau of Standards. US Government Printing Office, Washington, DC. 1970.

9. Smith, W. L., and Leadbetter, M. R., On the Renewal Function for the Weibull Distribution. Technometrics 5 (3), 393-396 (1963).

10. White, J. S., Weibull Renewal Analysis. Third Annual Aerospace Reliability and Maintainability Conference. pp 639-657. Society of Automotive Engineers, Inc. New York, New York. 1964.

Next page is blank.

DISTRIBUTION LIST

No. of Copies	Organization
12	Commander Defense Documentation Center ATTN: TIPCR Cameron Station Alexandria, VA 22314
1	Commander US Army Materiel Command ATTN: AMCDL 5001 Eisenhower Avenue Alexandria, VA 22304
1	Commander US Army Materiel Command ATTN: AMCMA 5001 Eisenhower Avenue Alexandria, VA 22304
	Commander US Army Materiel Command
1	ATTN: AMCRD
1	AMCRD-R
1	AMCRD-U
1	AMCRD-W
	5001 Eisenhower Avenue Alexandria, VA 22304
1	Commander US Army Materiel Command ATTN: AMCQA 5001 Eisenhower Avenue Alexandria, VA 22304
1	Commander US Army Materiel Command ATTN: AMCPA-SA 5001 Eisenhower Avenue Alexandria, VA 22304

No. of Copies	Organization
	Commander
	US Army Armament Command
1	ATTN: AMSAR-MAE
2	AMSAR-SAS
1	AMSAR-RD
1	AMSAR-QAR
1	AMCPM-CAWS
1	AMSAR-CP
	Rock Island, IL 61201
	Director, Army Materiel Systems Analysis Agency
	Aberdeen Research and Development Center
1	ATTN: AMXSY-E
1	AMXSY-M
1	AMXSY-R
	Aberdeen Proving Ground, MD 21005
	Commander
	US Army Aviation Systems Command
1	ATTN: AMSAV-D
	PO Box 209
	St. Louis, MO 63166
1	Commander
	US Army Ballistics Research Laboratory
	Aberdeen Proving Ground, MD 21005
1	Commander
	Harry Diamond Laboratories
	ATTN: Systems Analysis Office
	Washington, DC 20438
1	Commander
	US Army Electronics Command
	ATTN: AMSEL-SA
	Fort Monmouth, NJ 07703
1	Commander
	US Army Electronics Command
	ATTN: AMSEL-PL
	Fort Monmouth, NJ 07703
1	Commander
	US Army Mobility Command
	ATTN: AMSME-XS
	4300 Goodfellow Blvd.
	St. Louis, MO 63120

No. of Copies	Organization
1	Commander US Army Missile Command ATTN: AMSMI-DA Redstone Arsenal, AL 35869
	Commander US Army Tank-Automotive Command
1	ATTN: AMSTA-RPL (Tech Lib)
1	AMSTA-SR Warren, MI 48090
	Commander US Army Test and Evaluation Command
1	ATTN: AMSTE-SY
1	AMSTE-RA Aberdeen Proving Ground, MD 21005
1	Commander US Army Troop Support Command ATTN: Systems Analysis Office 4300 Goodfellow Blvd St. Louis, MO 63120
1	Commander US Army Safeguard Evaluation Agency ATTN: SSEA-T White Sands Missile Range, NM 88002
	Commander US Army Training and Doctrine Command
1	ATTN: ATCD-CI
1	ATCD-CF
1	ATCD-CP
1	ATCD-SI Fort Belvoir, VA 22060
1	Commander USA Combined Arms Combat Developments Activity ATTN: ATCACC Fort Leavenworth, KS 66027
	Commander Rock Island Arsenal Rodman Laboratories
1	ATTN: SARRI-LA
1	SARRI-LR
1	SARRI-LE Rock Island, IL 61201

No. of Copies	Organization
1	Commander Watervliet Arsenal ATTN: SWEWV-RDD Watervliet, NY 12189
1	Manager, Defense Logistics Information Exchange USA Logistics Management Center ATTN: Mr. Leon J. Scarbrough Fort Lee, VA 23801
  1 1	Director US Army Management Engineering Training Agency ATTN: AMXOM-AMS Library Rock Island, IL 61201
2	Commander US Army Logistics Management Center ATTN: AMXMC-LS Fort Lee, VA 23801
2	Commander US AMC Intern Training Center - USALMC ATTN: AMXRR-IT Texarkana, TX 75501
1	Commander Rock Island Arsenal ATTN: SARRI-LPL Rock Island, IL 61201